Exercise 12

Use the *noise terms phenomenon* to solve the following Volterra integral equations:

$$u(x) = -x + \tan x + \tan^2 x - \int_0^x u(t)\,dt$$

Solution

Assume that $u(x)$ can be decomposed into an infinite number of components.

$$u(x) = \sum_{n=0}^{\infty} u_n(x)$$

Substitute this series into the integral equation.

$$\begin{aligned}
\sum_{n=0}^{\infty} u_n(x) &= -x + \tan x + \tan^2 x - \int_0^x \sum_{n=0}^{\infty} u_n(t)\,dt \\
u_0(x) + u_1(x) + u_2(x) + \cdots &= -x + \tan x + \tan^2 x - \int_0^x [u_0(t) + u_1(t) + \cdots]\,dt \\
u_0(x) + u_1(x) + u_2(x) + \cdots &= \underbrace{-x + \tan x + \tan^2 x}_{u_0(x)} + \underbrace{\int_0^x [-u_0(t)]\,dt}_{u_1(x)} + \underbrace{\int_0^x [-u_1(t)]\,dt}_{u_2(x)} + \cdots
\end{aligned}$$

If we set $u_0(x)$ equal to the function outside the integral, then the rest of the components can be deduced in a recursive manner.

$$\begin{aligned}
u_0(x) &= -x + \tan x + \tan^2 x \\
u_1(x) &= \int_0^x [-u_0(t)]\,dt = -\int_0^x \left(-t + \tan t + \sec^2 t - 1\right) dt = x + \frac{1}{2}x^2 + \ln|\cos x| - \tan x \\
&\ \vdots
\end{aligned}$$

The noise terms, $\mp x$ and $\pm \tan x$, appear in both $u_0(x)$ and $u_1(x)$. Cancelling $-x$ and $\tan x$ from $u_0(x)$ leaves $\tan^2 x$. Now we check to see whether $u(x) = \tan^2 x$ satisfies the integral equation.

$$\begin{aligned}
\tan^2 x &\stackrel{?}{=} -x + \tan x + \tan^2 x - \int_0^x \tan^2 t\,dt \\
\tan^2 x &\stackrel{?}{=} \cancel{-x} + \cancel{\tan x} + \tan^2 x - (\cancel{\tan x} - \cancel{x}) \\
\tan^2 x &= \tan^2 x
\end{aligned}$$

Therefore,

$$u(x) = \tan^2 x.$$